Complete Classical Mechanics

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May 4, 2026

Contents

1 Survey of the Elementary Principles

1.1 Core ideas

Classical mechanics describes the motion of macroscopic bodies when quantum and relativistic effects are small. A particle is an idealized object with mass m and position $\boldsymbol{r}(t)$ in a three-dimensional Euclidean space. Its motion is described by velocity $\boldsymbol{v} = \dot{\boldsymbol{r}}$ and acceleration $\boldsymbol{a} = \ddot{\boldsymbol{r}}$.

Newton's Laws and Inertial Frames Newton's laws are valid only in **inertial frames**—frames where a body remains at rest or in uniform motion unless acted upon by a force.

1. **First Law (Inertia):** In an inertial frame, a particle moves with constant velocity if the net force is zero.
2. **Second Law (Dynamics):** The rate of change of momentum $\boldsymbol{p} = m\boldsymbol{v}$ equals the net force $\boldsymbol{F}$:
$$\boldsymbol{F} = \dot{\boldsymbol{p}} = \frac{d}{dt}(m\boldsymbol{v}).$$
For constant mass, this reduces to $\boldsymbol{F} = m\boldsymbol{a}$.
3. **Third Law (Action-Reaction):** For every force $\boldsymbol{F}_{12}$ exerted by particle 2 on particle 1, there is an equal and opposite force $\boldsymbol{F}_{21} = -\boldsymbol{F}_{12}$ exerted by particle 1 on particle 2. This holds for "central" forces acting along the line joining the particles.

Work and Energy The **work** done by a force $\boldsymbol{F}$ as a particle moves from point 1 to 2 is $W_{12} = \int_1^2 \boldsymbol{F} \cdot d\boldsymbol{r}$. The **Work-Energy Theorem** states that the work done by the net force equals the change in kinetic energy $T = \frac{1}{2}mv^2$:

$$W_{12} = T_2 - T_1 = \Delta T.$$

A force is **conservative** if the work done is independent of the path. Such forces can be derived from a potential energy $V(\boldsymbol{r})$:

$$\boldsymbol{F} = -\nabla V(\boldsymbol{r}) \implies W_{12} = V_1 - V_2 = -\Delta V.$$

In this case, total mechanical energy $E = T + V$ is conserved ($\Delta E = 0$).

Systems of Particles For a system of N particles, the total mass is $M = \sum m_i$ and the **Center of Mass (CM)** is $\boldsymbol{R} = \frac{1}{M}\sum m_i \boldsymbol{r}_i$.

- **Linear Momentum:** The total momentum $\boldsymbol{P} = \sum \boldsymbol{p}_i = M\dot{\boldsymbol{R}}$ changes according to the net external force: $\dot{\boldsymbol{P}} = \boldsymbol{F}_{\text{ext}}$. If $\boldsymbol{F}_{\text{ext}} = 0$, $\boldsymbol{P}$ is conserved.
- **Angular Momentum:** The total angular momentum $\boldsymbol{L} = \sum \boldsymbol{r}_i \times \boldsymbol{p}_i$ changes according to the net external torque $\boldsymbol{N}_{\text{ext}} = \sum \boldsymbol{r}_i \times \boldsymbol{F}_{i,\text{ext}}$: $\dot{\boldsymbol{L}} = \boldsymbol{N}_{\text{ext}}$.
- **Kinetic Energy:** $T = T_{\text{CM}} + T_{\text{rel}}$, where $T_{\text{CM}} = \frac{1}{2}M\dot{\boldsymbol{R}}^2$ is the kinetic energy of the CM motion and $T_{\text{rel}} = \frac{1}{2}\sum m_i(\dot{\boldsymbol{r}}_i')^2$ is the kinetic energy relative to the CM.

1.2 Mathematical spine

$$\begin{aligned}
\boldsymbol{F} &= \dot{\boldsymbol{p}} = m\boldsymbol{a} && \text{(Newton's 2nd Law)}\\
W_{12} &= \int_1^2 \boldsymbol{F}\cdot d\boldsymbol{r} = \Delta T && \text{(Work-Energy Theorem)}\\
\boldsymbol{L} &= \boldsymbol{r}\times\boldsymbol{p}, \quad \boldsymbol{N} = \boldsymbol{r}\times\boldsymbol{F} = \dot{\boldsymbol{L}} && \text{(Angular Momentum and Torque)}\\
M\ddot{\boldsymbol{R}} &= \boldsymbol{F}_{\text{ext}}, \quad \dot{\boldsymbol{L}} = \boldsymbol{N}_{\text{ext}} && \text{(System Dynamics)}
\end{aligned}$$

Example: Conservative vs. Non-conservative forces Gravity ($V = mgh$) and springs ($V = \frac{1}{2}kx^2$) are conservative. A conservative force does work that depends only on the endpoints; equivalently,

$$\oint \boldsymbol{F}\cdot d\boldsymbol{r} = 0$$

for every closed path. Kinetic friction violates this condition. Since

$$\boldsymbol{f}_k = -\mu N\hat{\boldsymbol{v}}, \qquad W_f = \int \boldsymbol{f}_k \cdot d\boldsymbol{r} = -\int \mu N\, ds,$$

the work depends on the arclength actually traveled, not just on the initial and final positions. For a closed loop of length $L > 0$ with constant μN,

$$W_f = -\mu N L < 0,$$

so friction cannot be represented by a single-valued potential energy. The heat produced is the physical destination of the lost mechanical energy; the mathematical reason friction is non-conservative is path dependence, or nonzero work around a closed path.

Section summary Newtonian mechanics builds from point particles to systems using three laws of motion. Conservation of energy, momentum, and angular momentum arise from symmetries and the nature of internal forces.

2 Variational Principles and Lagrange's Equations

2.1 Core ideas

Newtonian mechanics requires identifying all forces, including unknown constraint forces (like the normal force on a bead on a wire). The Lagrangian method bypasses these by using energy and **generalized coordinates** q_i.

Generalized Coordinates and Constraints For a system of N particles, there are $3N$ degrees of freedom. **Constraints** reduce this number.

- **Holonomic constraints** can be expressed as $f(r_1, r_2, \dots, t) = 0$. If there are k such constraints, the number of independent coordinates is $n = 3N - k$. These independent coordinates are the generalized coordinates $q_1, q_2, \dots, q_n$.
- Constraints that cannot be written this way are **non-holonomic**. Two common forms are inequalities such as $g(q,t) \leq 0$, and non-integrable velocity constraints

  $$\sum_i a_i(q,t)\dot{q}_i + a_0(q,t) = 0.$$

  If this differential relation can be integrated to $F(q,t) = 0$, it is holonomic; if it cannot, it is genuinely non-holonomic.

For example, a particle confined inside a box obeys $0 \leq x, y, z \leq L$, or inside a sphere obeys $x^2 + y^2 + z^2 \leq R^2$. These are inequality constraints rather than equations $F(q,t) = 0$. A more typical mechanics example is a car that rolls without sideways slipping. If (x, y) is the contact point and θ is the heading angle, the sideways velocity must vanish:

$$-\sin\theta\,\dot{x} + \cos\theta\,\dot{y} = 0.$$

This restricts allowed velocities but does not reduce to a position-only condition $F(x, y, \theta) = 0$. A rolling wheel gives similar non-integrable relations, such as $dx - R\cos\theta\,d\phi = 0$ and $dy - R\sin\theta\,d\phi = 0$.

Hamilton's Principle The Lagrangian is defined as $L(q, \dot{q}, t) = T - V$. Hamilton's Principle states that the actual path taken by a system between times t_1 and t_2 is the one that makes the **action** S stationary:

$$\delta S = \delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\,dt = 0.$$

This leads to the **Euler–Lagrange equations**:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, \dots, n.$$

The derivation is a direct integration-by-parts calculation. Compare the actual path $q_i(t)$ with a varied path $q_i(t) + \epsilon\eta_i(t)$, where the endpoint variations vanish: $\eta_i(t_1) = \eta_i(t_2) = 0$. Stationarity means

$$0 = \left.\frac{dS}{d\epsilon}\right|_{\epsilon=0} = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_i}\eta_i + \frac{\partial L}{\partial \dot{q}_i}\dot{\eta}_i\right) dt.$$

The second term is integrated by parts:

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_i}\dot{\eta}_i\,dt = \left[\frac{\partial L}{\partial \dot{q}_i}\eta_i\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\eta_i\,dt.$$

The boundary term is zero because the endpoints are fixed. Therefore

$$\delta S = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right]\eta_i\,dt = 0.$$

Since the functions $\eta_i(t)$ are arbitrary inside the interval, the bracket must vanish for each coordinate, giving the Euler–Lagrange equations.

Generalized Forces and Non-conservative Systems If some forces (like friction) are not derivable from a potential, the Euler–Lagrange equations are modified:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i,$$

where $Q_i = \sum_j \boldsymbol{F}_j^{\text{nc}} \cdot \frac{\partial \boldsymbol{r}_j}{\partial q_i}$ is the **generalized force** associated with coordinate q_i.

For a bead sliding on a circular hoop of radius R, with coordinate θ, the position is $\boldsymbol{r}(\theta) = R(\sin\theta, -\cos\theta)$ and $\partial \boldsymbol{r}/\partial\theta = R(\cos\theta, \sin\theta)$. If a tangential friction force has magnitude $bR\dot{\theta}$ and opposes the motion, then

$$\boldsymbol{F}^{\text{nc}} = -bR\dot{\theta}\,\hat{\boldsymbol{e}}_\theta, \qquad Q_\theta = \boldsymbol{F}^{\text{nc}} \cdot \frac{\partial \boldsymbol{r}}{\partial \theta} = -bR^2\dot{\theta}.$$

With $L = \frac{1}{2}mR^2\dot{\theta}^2 - mgR(1-\cos\theta)$, the equation of motion becomes

$$mR^2\ddot{\theta} + mgR\sin\theta = -bR^2\dot{\theta},$$

or

$$\ddot{\theta} + \frac{b}{m}\dot{\theta} + \frac{g}{R}\sin\theta = 0.$$

This example shows how non-conservative forces enter as generalized forces without being folded into V.

Symmetries and Noether's Theorem A coordinate q_j that does not appear in the Lagrangian ($\partial L/\partial q_j = 0$) is called **cyclic** or ignorable. Its conjugate momentum $p_j = \partial L/\partial \dot{q}_j$ is conserved:

$$\dot{p}_j = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j} = 0 \implies p_j = \text{const.}$$

Noether's Theorem generalizes this: every continuous symmetry of the Lagrangian corresponds to a conservation law.

- Time translation symmetry $\implies$ conservation of energy ($H = \sum \dot{q}_i p_i - L$).
- Spatial translation symmetry $\implies$ conservation of linear momentum.
- Rotational symmetry $\implies$ conservation of angular momentum.

2.2 Mathematical spine

$$L = T - V \qquad \text{(Lagrangian)}$$

$$\delta \int L dt = 0 \implies \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \qquad \text{(Euler–Lagrange)}$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \qquad \text{(Generalized Momentum)}$$

$$H = \sum \dot{q}_i p_i - L \qquad \text{(Hamiltonian/Energy)}$$

Example: Simple Pendulum Coordinate: angle θ. $L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 - mg(l - l\cos\theta)$. Euler–Lagrange: $\frac{d}{dt}(ml^2\dot{\theta}) - (-mgl\sin\theta) = 0 \implies \ddot{\theta} + \frac{g}{l}\sin\theta = 0$.

Section summary The Lagrangian formulation replaces force-based dynamics with an optimization principle. It automatically incorporates holonomic constraints and links symmetries to conservation laws via Noether's theorem.

3 The Central Force Problem

3.1 Core ideas

A central force points along the line joining two particles and depends only on their separation: $\boldsymbol{F} = F(r)\hat{\boldsymbol{r}}$. Examples include gravity and the Coulomb force.

Reduction to a One-Body Problem For two particles with masses m_1, m_2 and positions $\boldsymbol{r}_1, \boldsymbol{r}_2$, the Lagrangian is $L = \frac{1}{2}m_1\dot{\boldsymbol{r}}_1^2 + \frac{1}{2}m_2\dot{\boldsymbol{r}}_2^2 - V(|\boldsymbol{r}_1 - \boldsymbol{r}_2|)$. Transforming to the center of mass $\boldsymbol{R}$ and relative coordinate $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$:

$$L = \frac{1}{2}M\dot{\boldsymbol{R}}^2 + \frac{1}{2}\mu\dot{\boldsymbol{r}}^2 - V(r),$$

where $M = m_1 + m_2$ and $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the **reduced mass**. The CM moves at constant velocity, so we focus on the relative motion in the CM frame.

Conservation Laws Since the force is central, the torque is zero, and the **angular momentum $\boldsymbol{L} = \boldsymbol{r} \times \mu\dot{\boldsymbol{r}}$** is conserved. This implies the motion is confined to a plane. In plane polar coordinates (r, ϕ):

$$L_z = \mu r^2\dot{\phi} = \ell = \text{const.}$$

Energy is also conserved: $E = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) + V(r) = \text{const.}$

The Effective Potential Using $\dot{\phi} = \frac{\ell}{\mu r^2}$, we can write the radial energy equation:

$$E = \frac{1}{2}\mu\dot{r}^2 + V_{\text{eff}}(r), \quad V_{\text{eff}}(r) = V(r) + \frac{\ell^2}{2\mu r^2}.$$

The term $\frac{\ell^2}{2\mu r^2}$ is the **centrifugal barrier**. The radial motion is effectively one-dimensional.

The Orbit Equation Defining $u = 1/r$, the differential equation for the orbit shape is:

$$\frac{d^2u}{d\phi^2} + u = -\frac{\mu}{\ell^2 u^2}F(1/u).$$

For the inverse-square law $F(r) = -k/r^2$ (where $k = Gm_1m_2$), the solution is a conic section:

$$r(\phi) = \frac{\alpha}{1 + e\cos(\phi - \phi_0)}, \quad \alpha = \frac{\ell^2}{\mu k}, \quad e = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}.$$

The **eccentricity** e determines the orbit shape:

- $e = 0$: Circle ($E = V_{\text{eff,min}} = -\frac{\mu k^2}{2\ell^2}$)
- $0 < e < 1$: Ellipse ($V_{\text{eff,min}} < E < 0$)
- $e = 1$: Parabola ($E = 0$)
- $e > 1$: Hyperbola ($E > 0$)

Kepler's Laws

1. Planets move in elliptical orbits with the Sun at one focus.
2. A line joining a planet and the Sun sweeps out equal areas during equal intervals of time (equivalent to conservation of angular momentum).
3. The square of the orbital period T is proportional to the cube of the semi-major axis a: $T^2 = \frac{4\pi^2\mu}{k}a^3$.

The Runge-Lenz Vector For the $1/r$ potential, there is an additional conserved vector, the **Laplace–Runge–Lenz vector**:

$$\boldsymbol{A} = \boldsymbol{p} \times \boldsymbol{L} - \mu k \hat{\boldsymbol{r}}.$$

This vector points along the major axis and its conservation explains why orbits in a $1/r$ potential are closed and do not precess.

3.2 Mathematical spine

$$V_{\text{eff}}(r) = V(r) + \frac{\ell^2}{2\mu r^2} \qquad \text{(Effective Potential)}$$

$$\frac{d^2u}{d\phi^2} + u = \frac{\mu k}{\ell^2} \qquad \text{(Orbit Equation for } 1/r^2 \text{ force)}$$

$$r(\phi) = \frac{\alpha}{1 + e\cos\phi} \qquad \text{(Conic Section Solution)}$$

Example: Circular Orbit For a circular orbit at radius r_0, the force must equal the centripetal requirement: $|F(r_0)| = \frac{\mu v^2}{r_0} = \frac{\ell^2}{\mu r_0^3}$. This corresponds to the minimum of $V_{\text{eff}}(r)$.

Section summary The central force problem reduces to a 1D radial problem using conservation of angular momentum. For inverse-square forces, orbits are conic sections, consistent with Kepler's Laws.

4 The Kinematics of Rigid Body Motion

4.1 Core ideas

A rigid body is a system of particles where the distance between any two particles remains constant: $|\boldsymbol{r}_i - \boldsymbol{r}_j| = c_{ij}$. It has 6 degrees of freedom (3 for translation of a reference point, 3 for rotation).

Chasles' Theorem Any displacement of a rigid body can be decomposed into a translation of a chosen base point plus a rotation about an axis through that point. Typically, the Center of Mass (CM) is chosen as the base point.

Angular Velocity and Rotations The velocity of any point P in the body is $\boldsymbol{v} = \boldsymbol{V} + \boldsymbol{\omega} \times \boldsymbol{r}$, where $\boldsymbol{V}$ is the velocity of the base point and $\boldsymbol{\omega}$ is the **angular velocity** vector. Rotations are often described using **Euler angles** (ϕ, θ, ψ) which represent a sequence of three rotations (e.g., z-x'-z'' convention) to transform from a space-fixed frame to a body-fixed frame.

Why Euler angles in practice Euler angles provide the minimal set of three independent generalized coordinates for orientation, so that rotational dynamics can be cast directly into Lagrange's equations without redundant constraints. In the heavy symmetric top, for instance, the choice of (ϕ, θ, ψ) makes ϕ and ψ cyclic, immediately yielding two conserved momenta p_ϕ and p_ψ and reducing the problem to a one-dimensional motion in θ. The same coordinates are the standard *yaw–pitch–roll* variables used to describe spacecraft attitude, gyrocompasses, gimbal-mounted sensors, robotic end-effectors, and aircraft orientation.

In numerical practice one must remember the well-known **gimbal-lock** singularity at $\theta = 0, \pi$, where $\dot{\phi}$ and $\dot{\psi}$ become indistinguishable and the kinematic map $(\dot{\phi}, \dot{\theta}, \dot{\psi}) \to \boldsymbol{\omega}$ loses rank. For this reason attitude-control engineers often integrate the equivalent quaternion (Euler–Rodrigues) parameters and convert back to Euler angles only for human-readable output.

The Inertia Tensor The **inertia tensor** $\boldsymbol{I}$ is a symmetric 3×3 matrix that relates angular momentum $\boldsymbol{L}$ to angular velocity $\boldsymbol{\omega}$: $\boldsymbol{L} = \boldsymbol{I}\boldsymbol{\omega}$. Its components in a given basis are:

$$I_{ij} = \sum_n m_n (r_n^2 \delta_{ij} - x_{n,i} x_{n,j}).$$

- **Principal Axes:** For any point, there exists a set of axes where $\boldsymbol{I}$ is diagonal. The diagonal elements I_1, I_2, I_3 are the **principal moments of inertia**.
- **Parallel Axis Theorem:** If I_{cm} is the inertia tensor about the CM, the tensor about an axis parallel but shifted by $\boldsymbol{a}$ is $I = I_{\text{cm}} + M(a^2 \mathbf{1} - \boldsymbol{a} \otimes \boldsymbol{a})$.

4.2 Mathematical spine

$$\boldsymbol{v} = \boldsymbol{V} + \boldsymbol{\omega} \times \boldsymbol{r} \qquad \text{(Velocity in Rigid Body)}$$

$$\boldsymbol{L} = \boldsymbol{I}\boldsymbol{\omega}, \quad T_{\text{rot}} = \frac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{L} = \frac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{I}\boldsymbol{\omega} \qquad \text{(Angular Momentum and KE)}$$

$$I_{ij} = \int \rho(\boldsymbol{r})(r^2 \delta_{ij} - x_i x_j) d^3 r \qquad \text{(Continuum Inertia Tensor)}$$

Example: Inertia Tensor of a Cube For a uniform cube of side a and mass M about a corner, $I_{xx} = \frac{1}{3}Ma^2$, $I_{xy} = -\frac{1}{4}Ma^2$. About the center, the axes are principal and $I_{xx} = I_{yy} = I_{zz} = \frac{1}{6}Ma^2$.

Section summary Rigid body kinematics describes the 6-DOF motion through CM translation and rotation, characterized by the angular velocity vector and the inertia tensor.

5 The Rigid Body Equations of Motion

5.1 Core ideas

Dynamics in a rotating frame requires accounting for the rotation of the basis vectors. For any vector $\boldsymbol{A}$:

$$\left(\frac{d\boldsymbol{A}}{dt}\right)_{\text{space}} = \left(\frac{d\boldsymbol{A}}{dt}\right)_{\text{body}} + \boldsymbol{\omega} \times \boldsymbol{A}.$$

Euler's Equations Applying this to the angular momentum $\boldsymbol{L}$ in the body-fixed frame of principal axes:

$$\boldsymbol{N} = \dot{\boldsymbol{L}}_{\text{body}} + \boldsymbol{\omega} \times \boldsymbol{L}.$$

This gives **Euler's equations**:

$$\begin{aligned} I_1 \dot{\omega}_1 - (I_2 - I_3)\omega_2 \omega_3 &= N_1 \\ I_2 \dot{\omega}_2 - (I_3 - I_1)\omega_3 \omega_1 &= N_2 \\ I_3 \dot{\omega}_3 - (I_1 - I_2)\omega_1 \omega_2 &= N_3 \end{aligned}$$

Torque-Free Motion When $\boldsymbol{N} = 0$, both $E_{\text{rot}} = \frac{1}{2}\sum I_i \omega_i^2$ and $L^2 = \sum I_i^2 \omega_i^2$ are conserved.

- **Stability (Tennis Racket Theorem):** Rotation about the principal axes with the largest ($I_{\max}$) or smallest ($I_{\min}$) moments is stable. Rotation about the intermediate axis is unstable.
- **Symmetric Top:** If $I_1 = I_2 \neq I_3$, the angular velocity $\boldsymbol{\omega}$ precesses around the body-fixed symmetry axis z' with frequency $\Omega_{\text{body}} = \frac{I_3 - I_1}{I_1}\omega_3$.

The Heavy Symmetric Top A top with $I_1 = I_2$ spinning in a gravitational field exhibits complex motion:

- **Precession:** The symmetry axis rotates around the vertical (gravity) axis.
- **Nutation:** The symmetry axis bobs up and down between two polar angles θ_1 and θ_2.
- **Stability:** A "sleeping top" (vertical spin) is stable only if $\omega_3 > \frac{2}{I_3}\sqrt{MglI_1}$.

Nutation amplitude and frequency (fast-spin limit) For a top released with non-zero tilt θ_0 and large spin ω_3 the energy/angular-momentum conservation reduces $\theta(t)$ to motion in an effective potential. Linearising about the mean angle one finds small-amplitude nutation at the angular frequency

$$\omega_{\text{nut}} \;\approx\; \frac{I_3\,\omega_3}{I_1},$$

i.e. the nutation is fast compared with the precession $\omega_{\text{prec}} \approx Mgl/(I_3\omega_3)$, and the two satisfy the simple product

$$\omega_{\text{nut}}\,\omega_{\text{prec}} \;\approx\; \frac{Mgl}{I_1}.$$

The peak-to-peak amplitude of the wobble for a top released from rest at θ_0 is

$$\Delta\theta \;=\; \theta_2 - \theta_1 \;\approx\; \frac{2Mgl\sin\theta_0}{I_3\,\omega_3^2}\,\frac{I_1}{I_3},$$

which vanishes as ω_3^{-2} — a fast-spinning top exhibits steady precession with imperceptible nutation. The sleeping-top threshold above is the same statement at $\theta_0 = 0$: nutation about the vertical is bounded only when $I_3^2\omega_3^2 > 4MglI_1$.

5.2 Mathematical spine

$$I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 = N_1 \qquad \text{(Euler's Equations)}$$

$$\omega_{\text{prec}} = \frac{Mgl}{L} \qquad \text{(Slow Precession approximation)}$$

$$E = \frac{1}{2}I_1(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \frac{1}{2}I_3(\dot{\phi}\cos\theta + \dot{\psi})^2 + Mgl\cos\theta \qquad \text{(Heavy Top Energy)}$$

Example: Free Precession of the Earth The Earth is slightly oblate ($I_3 > I_1$). Euler's equations predict a precession of the spin axis (the "Chandler wobble") with a period related to the difference in moments of inertia.

Section summary Euler's equations describe rigid body dynamics in the body frame, revealing the stability of principal axis rotation and the precession/nutation of tops.

6 Oscillations

6.1 Core ideas

Most systems in classical mechanics behave like harmonic oscillators when displaced slightly from a stable equilibrium point.

Small Oscillations around Equilibrium Consider a system with generalized coordinates q_i and a potential $V(q_1, \dots, q_n)$. Equilibrium occurs where $\frac{\partial V}{\partial q_i} = 0$. If this is a minimum (all eigenvalues of the Hessian $\frac{\partial^2 V}{\partial q_i \partial q_j}$ are positive), the equilibrium is **stable**. Defining $\eta_i = q_i - q_{i0}$ as the displacement, the Lagrangian for small oscillations is:

$$L \approx \frac{1}{2} \sum_{i,j} (T_{ij} \dot{\eta}_i \dot{\eta}_j - V_{ij} \eta_i \eta_j),$$

where $V_{ij} = \left. \frac{\partial^2 V}{\partial q_i \partial q_j} \right|_0$ and T_{ij} is the kinetic energy matrix (usually constant for small displacements).

Normal Modes and the Secular Equation The equations of motion are $T\ddot{\boldsymbol{\eta}} + V\boldsymbol{\eta} = 0$. Assuming a sinusoidal solution $\boldsymbol{\eta}(t) = \boldsymbol{a} e^{i\omega t}$ leads to the generalized eigenvalue problem:

$$(V - \omega^2 T)\boldsymbol{a} = 0.$$

Nontrivial solutions exist only if the **secular equation** holds:

$$\det(V - \omega^2 T) = 0.$$

The roots ω_k^2 are the **normal frequencies**, and the corresponding vectors $\boldsymbol{a}_k$ define the **normal modes**.

Normal Coordinates There exists a linear transformation $\boldsymbol{\eta} = \boldsymbol{A}\boldsymbol{\zeta}$ that simultaneously diagonalizes T and V. The new coordinates ζ_k are **normal coordinates**, and each evolves independently as a simple harmonic oscillator: $\ddot{\zeta}_k + \omega_k^2 \zeta_k = 0$.

Damping and Driving Real systems have dissipation (Q-factor) and external forces.

- **Damping:** $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$. Solutions can be underdamped, overdamped, or critically damped.
- **Resonance:** When driven by $F_0 \cos \omega t$, the amplitude is maximized near $\omega \approx \omega_0$. The phase shift δ changes from 0 to π as ω passes through resonance.

6.2 Mathematical spine

$$\det(V - \omega^2 T) = 0 \qquad \text{(Secular Equation)}$$

$$L = \frac{1}{2}(\dot{\boldsymbol{\zeta}}^2 - \boldsymbol{\omega}^2 \boldsymbol{\zeta}^2) \qquad \text{(Lagrangian in Normal Coordinates)}$$

$$Q = \frac{\omega_0}{2\gamma} \qquad \text{(Quality Factor)}$$

Example: Two Coupled Pendulums For two identical pendulums coupled by a spring, there are two modes: 1. In-phase: $\omega_1 = \sqrt{g/l}$ (spring not stretched). 2. Out-of-phase: $\omega_2 = \sqrt{g/l + 2k/m}$ (spring acts as additional restoring force).

Section summary Oscillation theory linearizes motion near equilibrium, reducing complex coupled dynamics to independent normal modes via an eigenvalue problem.

7 The Classical Mechanics of the Special Theory of Relativity

7.1 Core ideas

Special Relativity modifies Newtonian mechanics for high speeds ($v \sim c$), based on the constancy of the speed of light c in all inertial frames.

Lorentz Transformations and Four-Vectors The transformation between two inertial frames moving at relative velocity v along x is:

$$x' = \gamma(x - vt), \quad t' = \gamma(t - vx/c^2), \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

Events are points in **Minkowski spacetime** described by four-vectors $x^\mu = (ct, \boldsymbol{r})$. The **invariant interval** is $ds^2 = c^2 dt^2 - d\boldsymbol{r}^2$.

Relativistic Dynamics The **proper time** τ is the time measured in the particle's rest frame: $d\tau = dt/\gamma$.

- **Four-velocity:** $u^\mu = \frac{dx^\mu}{d\tau} = \gamma(c, \boldsymbol{v})$.
- **Four-momentum:** $p^\mu = mu^\mu = (E/c, \boldsymbol{p})$, where $\boldsymbol{p} = \gamma m\boldsymbol{v}$ and $E = \gamma mc^2$.
- **Energy-Momentum Relation:** The norm of the four-momentum is invariant: $p^\mu p_\mu = (E/c)^2 - p^2 = m^2c^2$, leading to $E^2 = p^2c^2 + m^2c^4$.

Relativistic Lagrangian The action for a free particle is proportional to its proper time (the "longest" path in Minkowski space):

$$S = -mc \int ds = \int (-mc^2\sqrt{1 - v^2/c^2})dt.$$

The **free particle Lagrangian** is $L = -mc^2\sqrt{1 - v^2/c^2}$. For a particle in an electromagnetic field:

$$L = -mc^2\sqrt{1 - v^2/c^2} - q\phi + q\boldsymbol{A} \cdot \boldsymbol{v}.$$

7.2 Mathematical spine

$$p^\mu = (E/c, \boldsymbol{p}) \qquad \text{(Four-momentum)}$$

$$E^2 = p^2c^2 + m^2c^4 \qquad \text{(Energy-momentum relation)}$$

$$\boldsymbol{f} = \frac{d\boldsymbol{p}}{dt} = \frac{d}{dt}(\gamma m\boldsymbol{v}) \qquad \text{(Relativistic force/Minkowski force)}$$

Example: Relativistic Doppler Effect The frequency shift for a source moving away is $\nu = \nu_0\sqrt{\frac{1-v/c}{1+v/c}}$. Unlike the classical case, there is also a **transverse Doppler effect** ($\nu = \nu_0/\gamma$) due to time dilation.

Section summary Relativistic mechanics replaces absolute time with proper time and Newtonian momentum with four-momentum, unified by the invariant mass-shell condition.

8 The Hamilton Equations of Motion

8.1 Core ideas

Hamiltonian mechanics is a reformulation of classical mechanics that emphasizes the symmetry between coordinates q_i and momenta p_i. It describes motion as a flow in **phase space**.

The Legendre Transformation The Hamiltonian $H(q, p, t)$ is obtained from the Lagrangian $L(q, \dot{q}, t)$ via a **Legendre transformation**:

$$H(q, p, t) = \sum_i p_i \dot{q}_i - L(q, \dot{q}, t), \quad p_i = \frac{\partial L}{\partial \dot{q}_i}.$$

If $L = T - V$ and T is a homogeneous quadratic function of $\dot{q}$, then $H = T + V = E$ (the total energy).

Hamilton's Equations Taking the differential of H and using Euler–Lagrange equations leads to **Hamilton's canonical equations**:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

These are $2n$ first-order differential equations, unlike the n second-order Euler–Lagrange equations.

Poisson Brackets For any two functions $f(q, p, t)$ and $g(q, p, t)$ on phase space, the **Poisson bracket** is:

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right).$$

The time evolution of any function f is given by:

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}.$$

A quantity is conserved if $\{f, H\} = 0$ (and it has no explicit time dependence).

Phase Space and Liouville's Theorem A state is a point in the $2n$-dimensional phase space. **Liouville's Theorem** states that the density of points in phase space (or the volume of a region of points) is constant along the trajectories of the system.

8.2 Mathematical spine

$$H(q, p, t) = \sum p_i \dot{q}_i - L \qquad \text{(Hamiltonian)}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad \text{(Hamilton's Equations)}$$

$$\dot{f} = \{f, H\} + \frac{\partial f}{\partial t} \qquad \text{(Evolution equation)}$$

Example: Harmonic Oscillator in Phase Space $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$. Hamilton's equations: $\dot{q} = p/m$, $\dot{p} = -m\omega^2 q$. The trajectories in phase space are ellipses.

Section summary Hamiltonian mechanics treats coordinates and momenta as independent variables in phase space, providing a powerful framework for conservation laws and a bridge to quantum mechanics.

9 Canonical Transformations

9.1 Core ideas

A transformation from old coordinates (q, p) to new coordinates (Q, P) is **canonical** if it preserves the form of Hamilton's equations. This is equivalent to preserving the Poisson bracket relations:

$$\{Q_i, Q_j\}_{q,p} = 0, \quad \{P_i, P_j\}_{q,p} = 0, \quad \{Q_i, P_j\}_{q,p} = \delta_{ij}.$$

The Symplectic Condition Defining $\boldsymbol{\eta} = (q_1, \ldots, q_n, p_1, \ldots, p_n)^T$, Hamilton's equations can be written as $\dot{\boldsymbol{\eta}} = \boldsymbol{J}\nabla H$, where $\boldsymbol{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the **symplectic matrix**. A transformation $\boldsymbol{\eta} \to \boldsymbol{\xi}$ is canonical if the Jacobian $\boldsymbol{M} = \partial\boldsymbol{\xi}/\partial\boldsymbol{\eta}$ satisfies the **symplectic condition**:

$$\boldsymbol{M}\boldsymbol{J}\boldsymbol{M}^T = \boldsymbol{J}.$$

Generating Functions Canonical transformations are often derived from a **generating function**. There are four basic types depending on the choice of independent variables:

1. $F_1(q, Q, t)$: $p_i = \frac{\partial F_1}{\partial q_i}, \quad P_i = -\frac{\partial F_1}{\partial Q_i}$.
2. $F_2(q, P, t)$: $p_i = \frac{\partial F_2}{\partial q_i}, \quad Q_i = \frac{\partial F_2}{\partial P_i}$.
3. $F_3(p, Q, t)$: $q_i = -\frac{\partial F_3}{\partial p_i}, \quad P_i = -\frac{\partial F_3}{\partial Q_i}$.
4. $F_4(p, P, t)$: $q_i = -\frac{\partial F_4}{\partial p_i}, \quad Q_i = \frac{\partial F_4}{\partial P_i}$.

The new Hamiltonian K is related to the old H by $K = H + \frac{\partial F}{\partial t}$.

Goal of Canonical Transformations The primary use is to transform to a system where all coordinates are cyclic ($\partial K/\partial Q_i = 0$). In such a system, the new momenta P_i are constants of motion, and the problem is trivially solved.

9.2 Mathematical spine

$$\{Q_i, P_j\} = \delta_{ij} \qquad \text{(Preservation of Poisson Brackets)}$$
$$\boldsymbol{M}\boldsymbol{J}\boldsymbol{M}^T = \boldsymbol{J} \qquad \text{(Symplectic Condition)}$$
$$K = H + \frac{\partial F}{\partial t} \qquad \text{(Transformation of the Hamiltonian)}$$

Example: Identity Transformation The generating function $F_2(q, P) = \sum q_i P_i$ gives $p_i = P_i$ and $Q_i = q_i$, which is the identity transformation. Changing this to $F_2 = \sum f_i(q) P_i$ generates a point transformation.

Section summary Canonical transformations preserve the symplectic structure of phase space, allowing for coordinate changes that simplify the Hamiltonian and reveal constants of motion.

10 Hamilton–Jacobi Theory and Action-Angle Coordinates

10.1 Core ideas

Hamilton–Jacobi (HJ) theory is the ultimate canonical transformation: it seeks a transformation to a frame where the new Hamiltonian is zero, meaning all new coordinates and momenta are constants of motion.

The Hamilton–Jacobi Equation Using an $F_2(q, P, t)$ generating function, we call the result **Hamilton's Principal Function** $S(q, P, t)$. If the new Hamiltonian $K = H + \partial S/\partial t = 0$, then $P_i = \alpha_i$ (constants). The HJ equation is a first-order, non-linear partial differential equation for S:

$$H\left(q_1, \dots, q_n, \frac{\partial S}{\partial q_1}, \dots, \frac{\partial S}{\partial q_n}, t\right) + \frac{\partial S}{\partial t} = 0.$$

For time-independent Hamiltonians, we use **Hamilton's Characteristic Function** W, where $S(q, \alpha, t) = W(q, \alpha) - Et$. Then:

$$H\left(q, \frac{\partial W}{\partial q}\right) = E.$$

Separation of Variables The HJ equation is often solved by **separation of variables**: $W(q_1, \dots, q_n) = \sum W_i(q_i)$. This reduces the PDE to n independent ODEs, which can be solved by integration (quadratures).

Action–Angle Variables For periodic systems, **action–angle variables** (J, θ) are the most natural coordinates.

- **Action variable** $J_i = \frac{1}{2\pi} \oint p_i dq_i$, where the integral is over one cycle of the motion.
- **Angle variable** θ_i is the canonical conjugate to J_i.
- The Hamiltonian depends only on actions: $H = H(J_1, \dots, J_n)$.
- The equations of motion are trivial: $\dot{J}_i = 0$, $\dot{\theta}_i = \frac{\partial H}{\partial J_i} = \nu_i$ (the constant frequency).

10.2 Mathematical spine

$$H(q, \partial S/\partial q, t) + \partial S/\partial t = 0 \qquad \text{(Hamilton–Jacobi Equation)}$$

$$J_i = \frac{1}{2\pi} \oint p_i dq_i \qquad \text{(Action Variable)}$$

$$\theta_i(t) = \nu_i t + \beta_i \qquad \text{(Angle Variable evolution)}$$

Example: Harmonic Oscillator $H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 = E$. The action is $J = E/\omega$. The frequency is $\nu = \partial H/\partial J = \omega$. The angle θ is the phase of the oscillation.

Section summary Hamilton–Jacobi theory reduces dynamics to a single PDE, while action-angle variables provide the most efficient description for integrable periodic systems.

11 Classical Chaos

11.1 Core ideas

Chaos refers to the complex, unpredictable behavior that can arise in deterministic nonlinear dynamical systems. It is not due to noise or randomness but is an inherent property of the system's geometry.

Sensitivity to Initial Conditions The hallmark of chaos is that two trajectories starting very close together in phase space will diverge exponentially:

$$|\delta x(t)| \approx |\delta x(0)| e^{\lambda t},$$

where λ is the **Lyapunov exponent**. A positive λ implies that small uncertainties in measurement grow rapidly, making long-term prediction impossible.

Phase Space and Attractors

- **Phase Portrait:** A map of all possible states in phase space. Regular systems have trajectories on circles or tori.
- **Strange Attractor:** For dissipative chaotic systems, trajectories settle onto a complex, fractal-like structure in phase space.
- **Poincaré Section:** A way to simplify the analysis by taking a "snapshot" of the system's state each time it crosses a chosen surface in phase space. This reduces the continuous dynamics to a discrete **map**.

The KAM Theorem The **Kolmogorov–Arnold–Moser (KAM)** theorem addresses what happens to an integrable system when a small nonlinear perturbation is added. It states that many of the original "invariant tori" (regular orbits) survive the perturbation, but they are increasingly destroyed as the perturbation strength increases, leading to "stochastic" or chaotic regions.

Routes to Chaos Systems often become chaotic through a sequence of **bifurcations** as a parameter is varied. A common route is **period-doubling**, where the period of the oscillation doubles repeatedly until it becomes infinite (chaos).

11.2 Mathematical spine

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \ln \frac{|\delta x(t)|}{|\delta x(0)|} \qquad \text{(Lyapunov Exponent)}$$

$$x_{n+1} = r x_n (1 - x_n) \qquad \text{(Logistic Map - simplest chaos example)}$$

Concrete example: period-doubling cascade in the logistic map The discrete map $x_{n+1} = r x_n(1 - x_n)$ on $x \in [0, 1]$ shows the period-doubling route to chaos as the parameter r is increased:

- $r < 1$: the only stable fixed point is $x^* = 0$ (extinction).
- $1 < r < 3$: a single non-trivial fixed point $x^* = 1 - 1/r$ is stable.
- $r_1 = 3$: first period-doubling bifurcation; a stable 2-cycle appears.
- $r_2 \approx 3.4495$: bifurcation to a stable 4-cycle.
- $r_3 \approx 3.5441$: 8-cycle.
- $r_4 \approx 3.5644$: 16-cycle, and so on.
- $r_\infty \approx 3.56995$: accumulation point — onset of chaos.

The successive intervals shrink geometrically, and their ratios approach the universal **first Feigenbaum constant**

$$\delta \;=\; \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} \;=\; 4.669\,201\,609\ldots$$

The width of the bifurcating branches scales by a second universal constant $\alpha \approx 2.5029$. Feigenbaum showed in 1978 that δ and α are independent of the specific map: *any* smooth one-dimensional map with a single quadratic maximum — a driven pendulum, a forced oscillator, even a dripping faucet — displays the same numbers, providing the first experimentally verified case of universality in classical chaos.

Example: Driven Damped Pendulum A simple pendulum with friction and a periodic driving force. For small drive, it oscillates regularly. As the drive increases, it can undergo period-doubling and eventually move chaotically, never repeating its path.

Section summary Chaos demonstrates that deterministic laws can lead to unpredictable behavior through exponential sensitivity to initial conditions, often visualized through strange attractors and Poincaré sections.

12 Canonical Perturbation Theory

12.1 Core ideas

Many realistic systems are "nearly integrable," meaning they can be described as an integrable system plus a small perturbation parameterized by a dimensionless number $\epsilon \ll 1$:

$$H(J, \theta) = H_0(J) + \epsilon H_1(J, \theta),$$

where $J = (J_1, \ldots, J_n)$ and $\theta = (\theta_1, \ldots, \theta_n)$ are the action–angle variables of the unperturbed problem.

Time-independent perturbation theory The goal is to find a near-identity canonical transformation to new variables $(\bar{J}, \bar{\theta})$ such that the new Hamiltonian $K(\bar{J})$ depends only on $\bar{J}$. One uses a type-2 generating function

$$F_2(J, \bar{\theta}) = \sum_i J_i \bar{\theta}_i + \epsilon\, W(J, \bar{\theta}),$$

in which W is the unknown "correction" to the identity. Expanding order by order in ϵ and demanding that the angle-dependent pieces cancel, the first-order shift in the Hamiltonian is simply the angle-average of the perturbation,

$$K_1(\bar{J}) = \langle H_1(\bar{J}, \theta)\rangle_\theta = \frac{1}{(2\pi)^n} \int_0^{2\pi} H_1(\bar{J}, \theta)\, d^n\theta,$$

and the generating-function correction takes the Fourier form

$$W(\bar{J}, \bar{\theta}) = \sum_{\boldsymbol{n}\neq 0} \frac{i\, h_{\boldsymbol{n}}(\bar{J})}{\boldsymbol{n}\cdot\boldsymbol{\omega}(\bar{J})}\, e^{i\boldsymbol{n}\cdot\bar{\theta}}, \qquad \boldsymbol{\omega} = \partial H_0/\partial \bar{J},$$

where $h_{\boldsymbol{n}}$ are the Fourier coefficients of H_1 and $\boldsymbol{n} \in \mathbb{Z}^n \setminus \{0\}$.

12.2 Secular terms and the Lindstedt–Poincaré fix

Naive expansion in ϵ tends to produce *secular terms*: terms that grow without bound as a power of t and so spoil the validity of the series after a time of order $1/\epsilon$. The standard cure is the **Lindstedt–Poincaré method**, which absorbs the would-be secular contribution into a renormalization of the oscillation frequency.

Worked example: Duffing oscillator Consider the weakly anharmonic equation

$$\ddot{x} + x + \epsilon\, x^3 = 0, \qquad x(0) = A,\ \dot{x}(0) = 0.$$

A naive expansion $x = x_0 + \epsilon x_1 + \cdots$ gives $x_0 = A\cos t$ and an inhomogeneous equation for x_1 whose driving term contains $\cos t$, producing the resonant secular response

$$x_1(t) \supset -\tfrac{3}{8}A^3\, t \sin t,$$

which diverges with time. To remove it, rescale time by $\tau = \omega t$ with $\omega = 1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots$. The equation becomes

$$\omega^2 x'' + x + \epsilon x^3 = 0,$$

with $x' = dx/d\tau$. Choosing ω_1 to kill the resonant term gives the uniformly valid first-order approximation

$$\omega = 1 + \tfrac{3}{8}\epsilon A^2 + \mathcal{O}(\epsilon^2), \qquad x(t) \approx A\cos(\omega t).$$

The amplitude-dependent frequency shift is the hallmark of nonlinear oscillation.

12.3 Resonances and the pendulum normal form

Small denominators The denominators $\boldsymbol{n}\cdot\boldsymbol{\omega}$ in W become arbitrarily small whenever the unperturbed frequencies are nearly commensurate, $\boldsymbol{n}\cdot\boldsymbol{\omega} \approx 0$. The series then diverges — the celebrated "problem of small divisors" in celestial mechanics, partially tamed by KAM theory for sufficiently irrational frequency ratios.

Worked resonant example Consider two degrees of freedom near a $p:q$ resonance, $p\omega_1 - q\omega_2 \approx 0$, with perturbation

$$H = H_0(J_1, J_2) + \epsilon\, h\, \cos(p\theta_1 - q\theta_2).$$

Average over the *fast* (non-resonant) angles but retain the slow resonant angle $\psi = p\theta_1 - q\theta_2$. Introducing the canonical transformation generated by $F_2 = \psi P + \theta_2 P_2$ (so that $J_1 = pP$, $J_2 = -qP + P_2$) gives an effective one-degree-of-freedom Hamiltonian

$$H_{\text{eff}}(P, \psi) = H_0(pP, -qP + P_2) + \epsilon h \cos\psi.$$

Expanding H_0 about the resonant action P_* where $p\omega_1 = q\omega_2$, one obtains the **pendulum form**

$$H_{\text{eff}} \approx \tfrac{1}{2}M(P - P_*)^2 + \epsilon h\cos\psi, \qquad M \equiv \left.\frac{\partial^2 H_0}{\partial P^2}\right|_{P_*}.$$

Resonant motion thus librates like a pendulum with small-oscillation frequency $\Omega = \sqrt{|M|\,\epsilon h}$ and a separatrix of width $\Delta P \sim \sqrt{\epsilon h/|M|}$ — the characteristic "resonance island."

12.4 Adiabatic invariants

Concept If a parameter $\lambda(t)$ of the Hamiltonian changes on a timescale much longer than the orbital period T, i.e. $\dot{\lambda}/\lambda \ll 1/T$, then the action

$$J = \frac{1}{2\pi} \oint p\, dq$$

is conserved up to exponentially small corrections in the slowness parameter. J is therefore called an **adiabatic invariant**.

Worked example: harmonic oscillator with slowly varying frequency For $H = \frac{1}{2}p^2 + \frac{1}{2}\omega(t)^2 q^2$ the action of an instantaneous orbit at energy E is the area of a phase-space ellipse with semi-axes $\sqrt{2E}$ and $\sqrt{2E}/\omega$,

$$J = \frac{E}{\omega(t)}.$$

Adiabatic invariance of J therefore implies

$$E(t) = \omega(t)\, J = \omega(t)\, \frac{E_0}{\omega_0},$$

so the energy tracks the frequency: a pendulum whose string is slowly shortened gains energy in proportion to its raised frequency. Historically this is the result Einstein invoked at the 1911 Solvay conference and that Ehrenfest promoted to the quantization rule $J = n\hbar$ of the old quantum theory.

12.5 Mathematical spine

$$\begin{aligned}
H &= H_0(J) + \epsilon H_1(J, \theta) && \text{(perturbed Hamiltonian)} \\
K_1 &= \langle H_1 \rangle_\theta && \text{(first-order energy shift)} \\
\omega &= \omega_0 + \tfrac{3}{8}\epsilon A^2 + \cdots && \text{(Lindstedt frequency, Duffing)} \\
H_{\text{res}} &\approx \tfrac{1}{2}M\, \delta P^2 + \epsilon h \cos\psi && \text{(pendulum near resonance)} \\
J &= \tfrac{1}{2\pi} \oint p\, dq \approx \text{const} && \text{(adiabatic invariant)}
\end{aligned}$$

Astrophysical aside: precession of Mercury's perihelion General relativity adds an effective $1/r^3$ correction to the Newtonian $1/r$ potential. Treating it as ϵH_1 and applying the angle-average prescription yields the celebrated $43''$/century perihelion advance.

Section summary Canonical perturbation theory systematically organizes the effects of small departures from integrability. Secular terms in the naive series are removed by frequency renormalization (Lindstedt–Poincaré); near a resonance the same machinery reduces the dynamics locally to a pendulum; and adiabatic invariants such as E/ω for the slowly varying oscillator capture the robust quantities that survive slow parameter drift.

13 Introduction to the Lagrangian and Hamiltonian Formulations for Continuous Systems and Fields

13.1 Core ideas

Classical mechanics can be extended to systems with an infinite number of degrees of freedom, such as fluids, elastic solids, and electromagnetic fields. These are described by **fields** $\phi(x, y, z, t)$.

Lagrangian Density The total Lagrangian L is the spatial integral of a **Lagrangian density** $\mathcal{L}$:

$$L = \int \mathcal{L}(\phi, \partial_\mu \phi, x^\mu) d^3x.$$

The action is $S = \int L dt = \int \mathcal{L} d^4x$. Hamilton's Principle ($\delta S = 0$) leads to the **Euler–Lagrange equations for fields**:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0,$$

where $\partial_\mu = (\frac{1}{c}\partial_t, \nabla)$ is the four-gradient.

Hamiltonian Density The conjugate momentum density is $\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$. The Hamiltonian density is:

$$\mathcal{H} = \pi \dot{\phi} - \mathcal{L}.$$

The total Hamiltonian $H = \int \mathcal{H} d^3x$ gives the total energy of the field.

Noether's Theorem and the Stress-Energy Tensor Symmetries of the Lagrangian density lead to conserved currents.

- **Internal symmetries** lead to conserved charges (like electric charge).
- **Spacetime symmetries** lead to the conservation of the **Stress–Energy Tensor** $T^{\mu\nu}$:

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - g^{\mu\nu} \mathcal{L}.$$

  Conservation $\partial_\mu T^{\mu\nu} = 0$ implies conservation of energy and momentum.

13.2 Mathematical spine

$$\begin{aligned}
\delta \int \mathcal{L} d^4x = 0 \implies \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0 && \text{(Field Equations)} \\
\mathcal{H} = \pi \dot{\phi} - \mathcal{L} && \text{(Hamiltonian Density)} \\
T^{00} = \mathcal{H} && \text{(Energy Density)}
\end{aligned}$$

Example: The Vibrating String For a string with tension τ and linear density ρ, $\mathcal{L} = \frac{1}{2}\rho \dot{y}^2 - \frac{1}{2}\tau(\partial_x y)^2$. The Euler–Lagrange equation yields the wave equation: $\rho \ddot{y} - \tau y'' = 0$.

Example: The electromagnetic field The free Maxwell field is described by the gauge potential $A^\mu = (\phi/c, \boldsymbol{A})$ and the antisymmetric field-strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Its Lorentz-invariant Lagrangian density is

$$\mathcal{L}_{\text{EM}} = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} - J^\mu A_\mu,$$

with $J^\mu = (c\rho, \boldsymbol{J})$ the four-current. The Euler–Lagrange equations applied to A_μ reproduce the inhomogeneous **Maxwell equations** $\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$, while the homogeneous pair follows automatically from the antisymmetry of $F_{\mu\nu}$. The associated stress–energy tensor $T^{\mu\nu}$ reproduces the energy density $\frac{1}{2}(\varepsilon_0 E^2 + B^2/\mu_0)$ and the Poynting momentum density $\boldsymbol{E} \times \boldsymbol{B}/(\mu_0 c^2)$.

Example: Non-linear waves — the sine-Gordon kink A celebrated nonlinear field theory in 1+1 dimensions is the **sine-Gordon model** with Lagrangian density

$$\mathcal{L}_{\mathrm{sG}} \;=\; \tfrac{1}{2}(\partial_t\phi)^2 - \tfrac{1}{2}c^2(\partial_x\phi)^2 - \frac{m^2c^4}{\beta^2}\big[1-\cos(\beta\phi)\big],$$

which yields the equation of motion $\partial_t^2\phi - c^2\partial_x^2\phi + (m^2c^4/\beta)\sin(\beta\phi) = 0$. Despite being nonlinear, it admits an exact static **kink** (soliton) solution interpolating between adjacent vacua $\phi = 0$ and $\phi = 2\pi/\beta$:

$$\phi_K(x) \;=\; \frac{4}{\beta}\arctan\!\big[\exp(mc\,x)\big], \qquad E_{\mathrm{kink}} \;=\; \frac{8mc^3}{\beta^2}.$$

Boosting it gives a localised travelling wave that retains its shape after collisions — a hallmark of the integrability of the sine-Gordon system. The same equation describes mechanical chains of coupled pendulums and Josephson-junction transmission lines, illustrating how classical field theory unifies wave propagation in seemingly disparate physical systems.

Section summary Field theory generalizes discrete mechanics to continuous media, using Lagrangian densities and local field equations, providing the classical foundation for Electromagnetism and Quantum Field Theory.